

Mystery of Real Scalar Klein-Gordon Field

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The mystery that the real scalar Klein-Gordon field has vanishing current densities is resolved. The scalar field is shown to be a complex field due to the condition of possessing a proper non-relativistic limit. Like the Schrödinger field, one component complex Klein-Gordon field corresponds to a boson with one flavor, and therefore there exists no physical real scalar field. As a good example, we present the Schwinger model Hamiltonian which is naturally described by the complex scalar field with one flavor. The non-existence of the real scalar field indicates that the Higgs mechanism should be reconsidered.

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I. INTRODUCTION

The Klein-Gordon equation has a history of 80 years, and has been repeatedly discussed since it contains some basic problems. The Klein-Gordon equation can be written for the scalar field $\phi(x)$ with its mass m as

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi(x) = 0 \quad (1.1)$$

where we denote $x = (t, \mathbf{r})$. The scalar field ϕ has only one component, and therefore it is believed to describe the spinless particle. The Lagrangian density which reproduces eq.(1.1) can be written as

$$\mathcal{L} = \frac{1}{2} \left(\dot{\phi}^\dagger \dot{\phi} - \nabla \phi^\dagger \cdot \nabla \phi \right) - \frac{1}{2} m^2 \phi^\dagger \phi \quad (1.2)$$

which leads to the Hamiltonian density \mathcal{H}

$$\mathcal{H} = \frac{1}{2} \{ \Pi^\dagger \Pi + \nabla \phi^\dagger \cdot \nabla \phi + m^2 \phi^\dagger \phi \} \quad (1.3)$$

where $\Pi(x)$ is defined as $\Pi(x) = \dot{\phi}(x)$. In the present days, most of the field theory textbooks state that the real scalar field has vanishing current density, but if one quantizes the real scalar field, then it can be interpreted as the charge zero boson and therefore the zero current density problem may be resolved. Further, if the scalar field is complex, then it should correspond to the charged bosons like the π^\pm mesons. This is the standard understanding of the real scalar field.

However, there are basically two serious defects in the real scalar field solution in the Klein-Gordon equation. Zero current density and no non-relativistic limit. In this respect, the real scalar field is just like a ghost field, but it has been accepted as a physical particle and is indeed treated in most of the standard textbooks since it is simple in mathematics.

II. KLEIN-GORDON FIELD

Here, we examine whether a real scalar field can exist as a physical observable or not in the Klein-Gordon equation. Normally, we find that pion with the positive charge is an anti-particle of pion with the negative charge. This can be easily understood if we look into the structure of the pion in terms of quarks. π^\pm are indeed anti-particle to each other by changing quarks into anti-quarks. Since pion is not an elementary particle, their dynamics must be governed by the complicated quark dynamics. Under some drastic approximations, the motion of pion may be governed by the Klein-Gordon equation.

A. Physical Scalar Field

It looks that eq.(1.1) contains the negative energy state. However, one sees that eq.(1.1) is only one component equation and, therefore the eigenvalue of E^2 can be obtained as a physical observable. There is no information from the Klein-Gordon equation for the energy E itself, but only E^2 as we see it below,

$$(-\nabla^2 + m^2)\phi = E^2\phi. \quad (2.1)$$

In this case, the solution of eq.(2.1) should be described just in the same way as the Schrödinger field

$$\phi_{\mathbf{k}}(x) = A(t)e^{i\mathbf{k}\cdot\mathbf{r}} \quad (2.2)$$

which should be an eigenfunction of the momentum operator $\hat{\mathbf{p}} = -i\nabla$. The coefficient $A(t)$ can be determined by putting eq.(2.2) into eq.(1.1). One can easily find that $A(t)$ should be written as

$$A(t) = \frac{1}{\sqrt{V\omega_{\mathbf{k}}}} a_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}t} \quad (2.3)$$

where $a_{\mathbf{k}}$ is a constant, and $\omega_{\mathbf{k}}$ is given as

$$\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}. \quad (2.4)$$

The shape of $A(t)$ in eq.(2.3) can be also determined from the Lorentz invariance. Now, one sees that eq.(2.2) has a right non-relativistic limit. This is indeed a physical scalar field solution of the Klein-Gordon equation.

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B. Current Density

Now, we discuss the current density of the Klein-Gordon field which is obtained in terms of the Noether current as

$$\rho(x) = \frac{i}{2} \left(\phi^\dagger(x) \frac{\partial \phi(x)}{\partial t} - \frac{\partial \phi^\dagger(x)}{\partial t} \phi(x) \right) \quad (2.5)$$

$$\mathbf{j}(x) = -\frac{i}{2} [\phi^\dagger(x)(\nabla \phi(x)) - (\nabla \phi^\dagger(x))\phi(x)]. \quad (2.6)$$

1. Classical Real Scalar Field

The current density of a real scalar field vanishes to zero. This can be easily seen since the real scalar field $\phi(x)$ should satisfy

$$\phi^\dagger(x) = \phi(x). \quad (2.7)$$

Since the $\phi(x)$ is still a classical field, it is easy to see that the current densities of $\rho(x)$ and $\mathbf{j}(x)$ vanish to zero

$$\rho(x) = 0, \quad \mathbf{j}(x) = 0. \quad (2.8)$$

This means that there is no flow of the real scalar field, at least, classically. This is clear since a real wave function in the Schrödinger equation cannot propagate. Therefore, the condition that the scalar field should be a real field must be physically too strong. In the Schrödinger field, one cannot require that the field should be real. In fact, if one assumes that the field is real, then one obtains that the total energy of the Schrödinger field must vanish, and the field becomes unphysical [1].

Further, if the field is quantized, then the current density becomes infinity as shown in ref. [2].

2. Physical Scalar Field

The physical scalar field can be written as in eqs.(2.2) and (2.3)

$$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{V\omega_{\mathbf{k}}}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}}t}. \quad (2.9)$$

In this case, we can calculate the current density for the scalar field with the fixed momentum of \mathbf{k} and obtain

$$\rho(x) = \frac{|a_{\mathbf{k}}|^2}{V} \quad (2.10)$$

which is positive definite and finite. Therefore, the physical scalar field does not have any basic problems.

C. Composite Bosons

Pions and ρ -mesons are composed of quark and anti-quark fields. Suppressing the isospin variables, we can describe the boson fields in terms of the Dirac field $\psi_q(x)\chi_{\frac{1}{2}}$ only with the large components, for simplicity

$$\begin{aligned} \Psi^{(B)} &= \psi_q(x_1)\psi_{\bar{q}}(x_2)(\chi_{\frac{1}{2}}^{(1)} \otimes \chi_{\frac{1}{2}}^{(2)}) \\ &= \Phi^{(Rel)}(x)\Phi^{(CM)}(X)\xi_{s,s_z} \end{aligned} \quad (2.11)$$

where $x = x_1 - x_2$, $X = \frac{1}{2}(x_1 + x_2)$. Here, $\psi_{\bar{q}}(x)\chi_{\frac{1}{2}}$ denotes the anti-particle field and ξ_{s,s_z} is the spin wave function of the boson. $\Phi^{(Rel)}(x)$ denotes the internal structure of the boson and $\Phi^{(CM)}(X)$ corresponds to the boson field. Now, it is clear that the boson field $\Phi^{(CM)}(X)$ is a complex field.

In the field theory textbooks, the real scalar field is interpreted as a boson with zero charge. But this is not the right interpretation. The charge is a property of the field in units of the coupling constant. The positive and negative charges are connected to the flavor of the scalar fields. For example, a chargeless Schrödinger field, of course, has a finite current density of $\rho(\mathbf{r})$. The charge Q of the Schrödinger field is given as $Q = e_0 \int \rho(\mathbf{r})d^3r$ and for the chargeless field, we simply have $e_0 = 0$, which means that it does not interact with the electromagnetic field due to the absence of the coupling constant.

D. Gauge Field

The electromagnetic field A_μ is a real vector field which is required from the Maxwell equation, and therefore it has zero current density. However, the gauge field itself is gauge dependent and therefore it is not directly a physical observable. In this case, the energy flow in terms of the Poynting vector becomes a physical quantity. After the gauge fixing and the field quantization, the vector field $\hat{\mathbf{A}}(x)$ can be written as

$$\hat{\mathbf{A}}(x) = \sum_{\mathbf{k}} \sum_{\lambda=1}^2 \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \boldsymbol{\epsilon}(\mathbf{k}, \lambda) \left[c_{\mathbf{k},\lambda} e^{-i\mathbf{k}\cdot\mathbf{r}} + c_{\mathbf{k},\lambda}^\dagger e^{i\mathbf{k}\cdot\mathbf{r}} \right] \quad (2.12)$$

where $\omega_{\mathbf{k}} = |\mathbf{k}|$. Here, $\boldsymbol{\epsilon}(\mathbf{k}, \lambda)$ denotes the polarization vector. In this case, one-photon state with (\mathbf{k}, λ) becomes

$$\mathbf{A}_{\mathbf{k},\lambda}(x) = \langle \mathbf{k}, \lambda | \hat{\mathbf{A}}(x) | 0 \rangle = \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \boldsymbol{\epsilon}(\mathbf{k}, \lambda) e^{-i\mathbf{k}\cdot\mathbf{r} + i\omega_{\mathbf{k}}t}$$

which is the eigenstate of the momentum operator $\hat{\mathbf{p}} = -i\nabla$. In this respect, the gauge field A_μ is completely different from the Klein-Gordon scalar field. Naturally, the gauge field does not have any corresponding non-relativistic field.

III. BOSON IN THE SCHWINGER MODEL

There is a good example of the physical boson which is described in terms of the fermion operators in the Schwinger model. Since the boson in the Schwinger model is described in terms of the physical quantities, we can learn the essence of the physical boson how nature makes up a boson. Also, we learn the field quantization procedure of the boson fields.

A. Schwinger model

Here, we first describe briefly the Schwinger model which is a two dimensional QED with massless fermion. Its Lagrangian density can be written as [3]

$$\mathcal{L} = \bar{\psi} i \gamma_\mu D^\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (3.1)$$

where

$$D_\mu = \partial_\mu + ig A_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (3.2)$$

Here, A_μ denotes the vector potential in two dimensions. By taking the Coulomb gauge fixing $\partial^1 A_1 = 0$, we can describe the Hamiltonian of massless QED₂

$$H = \int \bar{\psi}(x) [-i\gamma^1 \partial_1 + g\gamma^1 A_1] \psi(x) dx + \frac{1}{2} \int \dot{A}_1^2 dx - \frac{g^2}{4} \int j_0(x) |x - x'| j_0(x') dx dx' \quad (3.3)$$

where the fermion current $j_\mu(x)$ is defined as

$$j_\mu(x) = \bar{\psi}(x) \gamma_\mu \psi(x).$$

The massless QED in two dimensions has a mass scale, that is, the coupling constant g has a mass dimension. Therefore, all the physical observables such as a boson mass are measured by the coupling constant g .

B. Bosonization

The Hamiltonian of the Schwinger model can be bosonized in the following fashion [4]

$$H = \frac{1}{2} \sum_p \left\{ \tilde{\Pi}^\dagger(p) \tilde{\Pi}(p) + p^2 \tilde{\Phi}^\dagger(p) \tilde{\Phi}(p) + \mathcal{M}^2 \tilde{\Phi}^\dagger(p) \tilde{\Phi}(p) \right\} \quad (3.4)$$

where the mass of the boson (Schwinger boson \mathcal{M}) is given as

$$\mathcal{M} = \frac{g}{\sqrt{\pi}}.$$

The boson fields $\tilde{\Phi}(p)$ and its conjugate field $\tilde{\Pi}(p)$ are related to the fermion current density in momentum representation $\tilde{j}_0(p)$ and $\tilde{j}_1(p)$ [1]

$$\tilde{j}_0(p) = ip \sqrt{\frac{4\pi}{L}} \tilde{\Phi}(p) = \int j_0(x) e^{ipx} dx \quad (3.5a)$$

$$\tilde{j}_1(p) = \sqrt{\frac{L}{\pi}} \tilde{\Pi}(p) = \int j_1(x) e^{ipx} dx \quad (3.5b)$$

where $\tilde{\Phi}(p)$ and $\tilde{\Pi}(p)$ satisfy the bosonic commutation relation

$$[\tilde{\Phi}(p), \tilde{\Pi}^\dagger(p')] = i\delta_{p,p'}. \quad (3.6)$$

This commutation relation is proved under the condition that the fermion operators $\tilde{j}_0(p)$ and $\tilde{j}_1(p)$ should operate on the physical states where, in the deep negative states, all the states are occupied while, in the highly excited states, there is no particle present.

For the zero mode, fields $\tilde{\Phi}(0)$ and its conjugate field $\tilde{\Pi}(0)$ are related to the regularized chiral charge and its time derivative, but we do not write them here since they are not relevant to the present discussion.

C. Complex field

The boson field $\tilde{\Phi}(p)$ is a complex function as one sees it from eq.(3.5) and can be written in terms of $\phi(x)$ in eq.(2.9)

$$\tilde{\Phi}(p) = \frac{1}{\sqrt{L}} \int \phi(x) e^{-ipx + i\omega_p t} dx = \frac{1}{\sqrt{\omega_p}} c_p. \quad (3.7)$$

Using $\Pi(x) = \dot{\phi}(x)$, we find

$$\tilde{\Pi}(p) = -i\sqrt{\omega_p} c_p. \quad (3.8)$$

In this case, one sees that the operators c_p, c_p^\dagger satisfy the right commutation relation due to eq.(3.6)

$$[c_p, c_q^\dagger] = \delta_{p,q}. \quad (3.9)$$

Therefore, we can write the Hamiltonian of eq.(3.4) as

$$H = \sum_p \omega_p c_p^\dagger c_p \quad (3.10)$$

which is just the proper expression of the boson Hamiltonian. It should be interesting to note that the boson Hamiltonian of eq.(3.10) has no zero point energy which is normally an infinite quantity. This must be related to the fact that the original Hamiltonian which is written in terms of the fermion operators does not contain any zero point energy. In fact, in the bosonization procedure, the vacuum energy as well as the charges of fermions are regularized in the gauge invariant fashion.

From this example of the physical boson field, we see that the scalar Klein-Gordon field must be described by the complex field which corresponds to one flavor boson. If one wishes to describe two charged bosons, then one has to introduce the isospin corresponding to a new degree of freedom. It is by now clear that there should not exist any real scalar field as a physical observable.

IV. HIGGS MECHANISM

If the scalar field is a complex function, then many properties of bosons are physically acceptable. In this case, however, there is a serious problem in connection with the Higgs mechanism. The Lagrangian density of the complex scalar field $\phi(x)$ which interacts with the $U(1)$ gauge field can be written as

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi)^\dagger(D^\mu\phi) - u_0(|\phi|^2 - \lambda^2)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (4.1)$$

where u_0 and λ are constant. In the Higgs mechanism, the scalar field is rewritten in terms of the two real scalar fields $\eta(x)$ and $\xi(x)$ as

$$\phi(x) = (\lambda + \eta(x)) e^{i\frac{\xi(x)}{\lambda}}. \quad (4.2)$$

After the spontaneous symmetry breaking, the $\eta(x)$ field remains physical, and the $\xi(x)$ field is absorbed into the gauge field which becomes massive by acquiring one degree of freedom. However, since the real scalar field is not physical, it is quite difficult to carry out the symmetry breaking mechanism in this fashion. At least, one cannot understand what the real scalar field indicates after the Higgs mechanism since the Lagrangian density contains the real scalar field of $\eta(x)$ after the spontaneous symmetry breaking.

In this respect, one should reexamine the Higgs mechanism from the point of view of the non-existence of the real scalar field. There is no doubt that the Weinberg-Salam model has achieved a great success for describing many experimental observations in connection with

the weak decays. Nevertheless, there are still some fundamental questions left for the basic mechanism of the spontaneous symmetry breaking on which the Weinberg-Salam model is entirely dependent.

V. DISCUSSIONS

The Klein-Gordon equation was discovered 80 years ago, and since then the boson is believed to be described by the Klein-Gordon equation. The scalar boson should exist if it is a composite object. In this case, one sees that the boson field should be complex like the Schrödinger field. In most of the field theory textbooks, however, it has been well accepted that the real scalar field should physically exist and that the real scalar field is described as the chargeless particle. At the same time, people realized that the real scalar field has some problems like vanishing current density. Therefore, it is stated in the textbooks that the real scalar field should be always quantized, and then it is all right.

Here, we point out that there should not exist any real scalar field. Instead, the scalar field must be always a complex function with only one component. In this case, one can define the finite current density and also the scalar field can possess the proper non-relativistic limit when the motion is slow. In addition, the bosonization procedure of the Schwinger model shows that the boson is a complex field with one flavor. Since the Schwinger boson is a physical object, it strongly suggests that the scalar field should be a complex field.

However, it is still not yet settled whether the scalar field or the Klein-Gordon equation itself for the elementary fields is physically acceptable or not. If the Klein-Gordon equation is fundamental, then it should have a proper degree of freedom which should be two while the Dirac equation has a correct degree of freedoms as a fermion field which is four.

The derivation of the Klein-Gordon equation is closely connected with the first quantization procedure which should be understood more in depth [1, 5].

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